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A THEORY OF LINEAR NON-CONSERVATIVE SYSTEMS*

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Linear systems with non-conservative positional forces are considered. It is proved that Rayleigh's theorem on the behaviour of the natural frequencies of conservative systems when the rigidity and inertia are varied cannot be generalized to such systems. A necessary and sufficient condition is established under which unstable non-conservative systems can be stabilized by dissipative forces of a special type.

It is shown that in the case of forced harmonic oscillations at frequencies lying beyond the spectrum of the corresponding conservative system, the application of non-conservative forces diminishes the absolute value of the action functional. Least upper bounds are obtained for the amplitudes of the forced oscillations, independent of the non-conservative forces.

1. The free oscillations of a system with non-conservative positional forces are described by the equation

$$Mx^{\cdot\cdot} + Ax = 0 \tag{1.1}$$
$$A = C + K, \quad M = \| m_{ij} \|_1^n, \quad C = \| c_{ij} \|_1^n, \quad K = \| k_{ij} \|_1^n,$$
$$\mathbf{x} = (x_1, \ldots, x_n)^T$$

where $\mathbf{x}$ is the vector of generalized coordinates, M and C are the symmetric inertia and elasticity matrices and K is the skew-symmetric matrix of non-conservative forces.

By Rayleigh's Theorem /1/, the frequencies of the natural oscillations of the corresponding conservative system $(K = 0)$ increase (do not decrease) as the rigidity increases and as the inertia of the system decreases. Zhuravlev has generalized this theorem to systems with gyroscopic forces /2/. He has suggested the following problem: is the analogous propostion true for system (1.1) when the non-conservative forces are sufficiently small? Below we shall answer this question in the negative.

We may assume without loss of generality that $M = E$ is the unit matrix. Let λ_i be a simple real eigenvalue of A, $\mathbf{a}_i$ a corresponding eigenvector and $\mathbf{b}_i$ an eigenvector of the transposed matrix A^T corresponding to λ_i. In general, the vectors $\mathbf{a}_i$ and $\mathbf{b}_i$ are linearly independent; we shall assume henceforth that this is indeed the case. Since $(\mathbf{a}_i, \mathbf{b}_i) \neq 0$ (where the parentheses denote the scalar product) /3/, we may assume that $(\mathbf{a}_i, \mathbf{b}_i) = 1$.

Put $C(\varepsilon) = C_0 + \varepsilon C_1$ in (1.1), where C_1 is a symmetric positive definite matrix. Let us investigate the behaviour of $\lambda_i(\varepsilon)$ as ε increased. We shall show that, unlike the conservative case, $\lambda_i(\varepsilon)$ is a decreasing function of ε when C_1 is suitably chosen.

As we know,

$$\delta_i = d\lambda_i(\varepsilon)/d\varepsilon\,|_{\varepsilon=0} = (\mathbf{a}_i, C_1\mathbf{b}_i) \tag{1.2}$$

Putting $\mathbf{a}_i = \mathbf{c}_i + \mathbf{d}_i$, $\mathbf{b}_i = \mathbf{c}_i - \mathbf{d}_i$ and using the symmetry of C_1, we obtain

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$$\delta_i = (\mathbf{c}_i, C_1\mathbf{c}_i) - (\mathbf{d}_i, C_1\mathbf{d}_i) \tag{1.3}$$

Let N be an orthogonal matrix whose first row is $\mathbf{c}_i/\|\mathbf{c}_i\|$ and Δ the diagonal matrix with elements $0,1,\ldots,1$; $S = N^T\Delta N$. It is obvious that S is non-negative definite; it has a simple eigenvalue $\lambda = 0$ with eigenvector $\mathbf{c}_i$; the other eigenvalues are 1. Therefore $(\mathbf{c}_i, S\mathbf{c}_i) = 0$; $(\mathbf{d}_i, S\mathbf{d}_i) > 0$ because $\mathbf{c}_i$ and $\mathbf{d}_i$ are linearly independent.

Put $C_1 = S + \mu E$; clearly, if $\mu > 0$ the matrix C_1 is positive definite. Since $\delta_i < 0$ when $\mu = 0$, this inequality is also true for sufficiently small μ. Thus, with this choice of C_1, the eigenvalue $\lambda_i(\varepsilon)$ is a decreasing function of ε in a certain interval $(0, \varepsilon_*)$, irrespective of the fact that $C(\varepsilon)$ is an increasing function.

Similarly one shows that, unlike the conservative case, λ_i may decrease as the inertia matrix M decreases.

As an illustrative example, consider

$$M = E,\quad C = \begin{Vmatrix} 1 & 0 \\ 0 & 3 \end{Vmatrix},\quad K = \begin{Vmatrix} 0 & \delta \\ -\delta & 0 \end{Vmatrix},\quad \delta^2 = \frac{3}{4}$$

The first eigenvalue of $A = C + K$ is $\lambda_1 = 1.5$; corresponding eigenvectors of A at A^T are $\mathbf{a}_1 = (\sqrt{2}\delta, \sqrt{2}/2)$, $\mathbf{b}_1 = (\sqrt{2}\delta, -\sqrt{2}/2)$. Therefore $\mathbf{c}_1 = (\sqrt{2}\delta, 0)$, $\mathbf{d}_1 = (0, \sqrt{2}/2)$, $N = E$, $S = \mathrm{diag}\,(0, 1)$. Put $C_1 = \mathrm{diag}\,(\mu, 1)$; then the eigenvalues $\lambda_i(\varepsilon)$ are the roots of the equation

$$F(\lambda, \varepsilon) = \lambda^2 - (4 + \varepsilon + \mu\varepsilon)\lambda + 3 + \varepsilon + \delta^2 + 3\mu\varepsilon + \mu\varepsilon^2 = 0 \tag{1.4}$$

From this equation we obtain $\delta_1 = {}^3/_2\mu - {}^1/_2$; hence, in the interval $\mu \in (0, {}^1/_3)$ $\lambda_1(\varepsilon)$ decreases for small ε, although $C(\varepsilon)$ increases.

Note that the proposition just proved by no means implies that one can reduce λ_i to any given value by suitably increasing C. As we know /4/, the real parts of the eigenvalues of the matrix A satisfy the inequality

$$\nu_1 \leqslant \mathrm{Re}\lambda_i \leqslant \nu_n \tag{1.5}$$

where ν_1 is the least and ν_n the greatest eigenvalue of C. Since when C is increased so is ν_1, it follows that the real eigenvalues of A also increase "on the whole", though they may decrease in certain intervals.

Remark. A typical situation in the stability analysis of non-conservative mechanical systems (1.1) is the following. The elements of the matrix A depend on some parameter μ (in aero-elasticity problems μ is the wind flow velocity /5/). For $\mu = 0$ the eigenvalues of A are real and positive, i.e., system (1.1) is stable. The critical value of μ is the value at which at least one characteristic exponent crosses over into the right half-plane. This occurs when positive eigenvalues λ_i and λ_{i+1} meet (oscillatory loss of stability - flutter) or when the sign of the least eigenvalue λ_1 changes (non-oscillatory loss of stability - divergence). It follows from inequality (1.5) that if the corresponding conservative system $(K = 0)$ remains stable as μ increases, then divergence is impossible.

2. We now consider stabilization of an unstable non-conservative system by means of dissipative forces. It is well-known /5/ that the application of such forces may also, generally speaking, destabilize a stable non-conservative system. We shall therefore confine our attention to a special class of dissipative forces, for which the matrix of the dissipation coefficients is proportional to the inertia matrix (damping of this type is usually referred to as external /6/). The corresponding equation of motion may be written as

$$\mathbf{x}^{\cdot\cdot} + \varepsilon\mathbf{x}^{\cdot} + A\mathbf{x} = 0 \tag{2.1}$$

where the positive parameter ε characterizes the magnitude of the dissipative forces.

Let $\lambda_j = a_j + ib_j$ $(j = 1,\ldots,n)$ be the eigenvalues of A. If one of these eigenvalues has $b_j \neq 0$ or $a_j < 0$, then system (2.1) is unstable at $\varepsilon = 0$.

Lemma 1. System (2.1) is asymptotically stable if and only if

$$\varepsilon^2 > b_j^2/a_j,\quad j = 1,\ldots,n \tag{2.2}$$

Proof. The change of variables $\mathbf{x} = \mathbf{y}\exp(-{}^1/_2\varepsilon t)$ reduces Eq.(2.1) to the form

$$\mathbf{y}^{\cdot\cdot} + (A - {}^1/_4\varepsilon^2 E)\,\mathbf{y} = 0 \tag{2.3}$$

Putting $\mathbf{y} = \mathbf{y}_j\exp(c_j + id_j)\,t$, we find that the characteristic exponents of system (2.3) satisfy the equality $(c_j + id_j)^2 = {}^1/_4\varepsilon^2 - \lambda$. Therefore,

$$c_j^{1,2} = \pm\{{}^1/_2[{}^1/_4\varepsilon^2 - a_j + (({}^1/_4\varepsilon^2 - a_j)^2 + b_j^2)^{1/2}]\}^{1/2} \tag{2.4}$$

It is clear that system (2.1) is asymptotically stable if $\varepsilon/2 > |c_j^{1,2}|$ for all j. In view of (2.4), this condition implies inequality (2.2).

As follows from (2.2), a non-conservative system is stabilized by dissipative forces of the indicated type provided $a_j > 0$ for all j. In view of inequality (1.5), we see that if the corresponding conservative system is stable, then for any non-conservative forces system (2.1) is asymtotically stable from some value of ε onwards.

The quantities b_i satisfy the inequality /4/

$$b_i \leqslant k_* \sqrt{{}^1/_2 n\,(n-1)}, \quad k_* = \max k_{ij} \tag{2.5}$$

In view of (1.5) and (2.5), we obtain the sufficient condition for the asymptotic stability of system (2.1) which does not involve evaluating the eigenvalue of A:

$$\varepsilon^2 > {}^1/_2 k_*^2 n\,(n-1)/\nu_1 \tag{2.6}$$

This condition is most useful when one has only an upper limit for the elements of the matrix of non-conservative forces.

3. We now consider forced harmonic oscillations in a system with non-conservative positional forces:

$$\mathbf{x}^{\cdot\cdot} + C\mathbf{x} + K\mathbf{x} = \mathbf{p}\cos\omega t, \quad \mathbf{p} = (p_1, \ldots, p_n)^T \tag{3.1}$$

Let us assume that the frequence of the applied force lies outside the frequency range of the corresponding conservative system, i.e., $\omega^2 \not\equiv [\nu_1, \nu_n]$. Then by (1.5), $\det|C + K - \omega^2 E| \neq 0$, and so Eq.(3.1) has a periodic solution $\mathbf{x} = \mathbf{a}\cos\omega t$, where the vector $\mathbf{a}$ is determined from the equation

$$(C + K - \omega^2 E)\,\mathbf{a} = \mathbf{p} \tag{3.2}$$

Put

$$J = \frac{1}{2}\int_0^{2\pi/\omega} ((\mathbf{x}^\cdot, \mathbf{x}^\cdot) - (C\mathbf{x}, \mathbf{x}) + (2\mathbf{p}\cos\omega t, \mathbf{x}))\,dt \tag{3.3}$$

The quantity J is the action integral, evaluated over an interval of length equal to the oscillation period. For a periodic solution, we have

$$J = {}^1/_2\pi\omega^{-1}(\omega^2(\mathbf{a}, \mathbf{a}) - (C\mathbf{a}, \mathbf{a}) + 2(\mathbf{p}, \mathbf{a})) \tag{3.4}$$

For fixed C and $\mathbf{p}$ the action J is a function of the elements k_{ij} of the matrix K, i.e., $J = J(K)$.

Lemma. $|J(K)|$ reaches its maximum value at $K = 0$.

Proof. By (3.2), $(C\mathbf{a}, \mathbf{a}) - \omega^2(\mathbf{a}, \mathbf{a}) = (\mathbf{p}, \mathbf{a})$, and therefore

$$J = {}^1/_2\pi\omega^{-1}(R\mathbf{a}, \mathbf{a}), \quad R = C - \omega^2 E \tag{3.5}$$

Noting that $(R^{-1})^T = R^{-1}$, $(K, \mathbf{a}, \mathbf{a}) = -(\mathbf{a}, K\mathbf{a})$, we infer from (3.2) that

$$\begin{aligned}(R^{-1}\mathbf{p}, \mathbf{p}) &= (R^{-1}(R + K)\,\mathbf{a}, (R + K)\,\mathbf{a}) = \\ &(\mathbf{a}, R\mathbf{a}) + (R^{-1}K\mathbf{a}, K\mathbf{a})\end{aligned} \tag{3.6}$$

If $\omega^2 < \nu_1$, then $R > 0$, and therefore $J(K) > 0$, $(R^{-1}K\mathbf{a}, K\mathbf{a}) \geqslant 0$, and by (3.6) we have $J \leqslant {}^1/_2\pi\omega^{-1}(R^{-1}\mathbf{p}, \mathbf{p}) = J(0)$. If $\omega^2 > \nu_n$, then $R < 0$, $J(K) < 0$, $J \geqslant {}^1/_2\pi\omega^{-1}(R^{-1}\mathbf{p}, \mathbf{p}) = J(0)$. Consequently, in either case $|J(K)| \leqslant |J(0)|$.

Thus, application of non-conservative forces reduces the absolute value of the action functional.

If $K \neq 0$ one has $J(K) = J(0)$, provided $K\mathbf{a} = 0$. In that situation, as one sees from (3.2), the amplitudes of the conservative and non-conservative systems are the same.

The inequality $|(\mathbf{a}, R\mathbf{a})| \leqslant |R^{-1}\mathbf{p}, \mathbf{p}|$, valid for any matrix K, implies that the amplitudes $|a_i|$ of the oscillations of system (3.1) bounded above by quantities independent of the magnitudes of the non-conservative forces. It is therefore important to determine the maximum of $|a_i|$ relative to the set of all skew-symmetric matrices K.

Let R_i, K_i denote the matrices obtained from R and K by deleting their i-th rows and columns, $V_i = R_i + K_i$; $\mathbf{r}_i, \mathbf{k}_i, \mathbf{a}_*$ and $\mathbf{p}_*$ the vectors formed from the i-th columns of R_i, K_i and the vectors $\mathbf{a}, \mathbf{p}$ by deleting the i-th components.

Lemma 3. The maximum of the i-th oscillation amplitude of system (3.1) is

$$\begin{gathered}a_i^* = |c_i| + \sqrt{c_i^2 + d_i} \\ c_i = \frac{p_i - \mathbf{r}_i^T R_i^{-1}\mathbf{p}_*}{2(r_{ii} - \mathbf{r}_i^T R_i^{-1}\mathbf{r}_i)}, \quad d_i = \frac{\mathbf{p}_*^T R_i^{-1}\mathbf{p}_*}{4(r_{ii} - \mathbf{r}_i^T R_i^{-1}\mathbf{r}_i)}\end{gathered} \tag{3.7}$$

The equality $|a_i| = a_i^*$ holds when the elements of K satisfy (3.11).

Proof. Let us first assume that the elements of K_i are fixed and determine the maximum of $|a_i|$ as a function of the elements of the vector $\mathbf{k}_i$.

We will first show that the maximum indeed exists.

The determinant of the matrix $R+K$ may be written as

$$\Delta = \Delta_0 + (\mathbf{k}_i, \mathbf{c}) + (\mathbf{k}_i, S\mathbf{k}_i)$$

where Δ_0 is the determinant of the matrix R, $(\mathbf{k}_i, \mathbf{c})$ is a linear and $(\mathbf{k}_i, S\mathbf{k}_i)$ a quadratic form in the coefficients k_{ij}. We assert that S is of fixed sign - in fact, of the same sign as R ($S>0$ if $\omega^2 < \nu_1$, $S<0$ if $\omega^2 > \nu_n$). Indeed, if $S \not> 0$ for $R>0$ or $S \not< 0$ for $R<0$, then $\Delta = 0$ for some $\mathbf{k}_i$, and so the matrix $R+K$ has an eigenvalue $\lambda_i = 0$. But this is impossible, since $\nu_1 - \omega^2 \leqslant \operatorname{Re}\lambda_i \leqslant \nu_i - \omega^2$. Assume now that $(\mathbf{k}_i, \mathbf{a}) = 0$ and $(\mathbf{k}_i, S\mathbf{k}_i) = 0$ for some $\mathbf{k}_i = \mathbf{k}_i^*$. Putting $\mathbf{k}_i = \varepsilon\mathbf{k}_i^*$ and expressing the solution of (3.2) as $a_i = \Delta_i/\Delta$, we find that Δ_i is a linear function of ε (if necessary, one can apply a small perturbation to the vector $\mathbf{p}$ to ensure that the coefficient of ε not vanish), whereas Δ is independent of ε. Therefore $|a_i| \to \infty$ as $\varepsilon \to \infty$, contrary to the boundedness of $|a_i|$. Thus S is of fixed sign; consequently, $|a_i| \to 0$ as $\|\mathbf{k}_i\| \to \infty$. Thus the supremum of $|a_i(\mathbf{k}_i)|$ is achieved at finite values of $\mathbf{k}_i$, i.e., the required maximum of $|a_i(\mathbf{k}_i)|$ exists.

We write system (3.2) as

$$a_i r_{ii} + ((\mathbf{r}_i - \mathbf{k}_i), \mathbf{a}_*) = p_i, \quad a_i(\mathbf{r}_i + \mathbf{k}_i) + V_i\mathbf{a}_* = \mathbf{p}_* \tag{3.8}$$

Expressing $\mathbf{a}_*$ in the second equation of (3.8) by means of the inverse matrix V^{-1}, we infer from the first equation that

$$a_i = \frac{p_i - (\mathbf{r}_i^T - \mathbf{k}_i^T)V_i^{-1}\mathbf{p}_*}{r_{ii} - \mathbf{r}_i^T V_i^{-1}\mathbf{k}_i + \mathbf{k}_i^T V_i^{-1}\mathbf{k}_i - \mathbf{r}_i^T V_i^{-1}\mathbf{r}_i + \mathbf{k}_i^T V_i^{-1}\mathbf{r}_i} \tag{3.9}$$

At the maximum of $|a_i(\mathbf{k}_i)|$ the derivatives of (3.9) with respect to the components of $\mathbf{k}_i$ must vanish. This yields the following system of equations in $\mathbf{k}_i$:

$$-(V_i^{-1} + (V_i^{-1})^T)\,\mathbf{k}_i = (V_i^{-1} + (V_i^{-1})^T)\,\mathbf{r}_i - a_i^{-1}V_i^{-1}\mathbf{p}_* \tag{3.10}$$

The inverse of $V_i^{-1} + (V_i^{-1})^T$ is $^1/_2 V_i^T R^{-1} V_i$. Indeed, using the fact that $V_i + V_i^T = 2R_i$, we obtain

$$V_i^T R_i^{-1} V_i (V_i^{-1} + (V_i^{-1})^T) = V_i^T R_i^{-1} + V_i^T R_i^{-1}(2R - V_i^T)(V_i^{-1})^T = 2E$$

Hence the solution of system (3.10) is

$$\begin{aligned}\mathbf{k}_i &= -{}^1/_2 V_i^T R_i^{-1} V_i\,((V_i^{-1} - (V_i^{-1})^T)\,\mathbf{r}_i - a_i^{-1}V_i^{-1}\mathbf{p}_* = \\ &{}^1/_2 a_i^{-1} V_i^T R_i^{-1}\mathbf{p}_* - {}^1/_2 V_i^T R_i \mathbf{r}_i + {}^1/_2 V_i^T R_i^{-1}(2R_i - V_i^T)(V_i^{-1})^T\mathbf{r}_i = \\ &{}^1/_2 a_i^{-1} V_i^T R_i^{-1}\mathbf{p}_* + \mathbf{r}_i - V_i^T R_i^{-1}\mathbf{r}_i\end{aligned} \tag{3.11}$$

Substituting this expression into (3.9) and using the fact that

$$(R_i^{-1})^T = R_i^{-1}, \quad R_i^T = R_i, \quad K_i^T = -K_i, \quad \mathbf{a}^T B\mathbf{c} = \mathbf{c}^T B^T \mathbf{a}$$

for any $\mathbf{a}$, B, $\mathbf{c}$, we obtain, after some reduction,

$$a_i = [p_i + {}^1/_2 a_i^{-1}\mathbf{p}_*^T R_i^{-1}\mathbf{p}_* - \mathbf{r}_i^T R_i^{-1}\mathbf{p}_*]\,[r_{ii} + {}^1/_4 a_i^{-2}\mathbf{p}_*^T R_i^{-1}\mathbf{p}_* - \mathbf{r}_i^T R^{-1}\mathbf{r}_i]^{-1}$$

Thus the stationary value of $a_i(\mathbf{k}_i)$ is a root of the quadratic equation

$$a_i^2 - 2c_i a_i - d_i = 0 \tag{3.12}$$

The largest absolute value of a root of Eq.(3.12), a_i^*, equals the maximum of $|a_i|$ as a function of the variables k_{ij} ($j = 1, \ldots, n$; $j \neq i$). But since the coefficients of this equation are independent of the elements of K_i, it follows that a_i^* equals the maximum of $|a_i|$ as a function of all the variables k_{ij}, i.e., it is the required maximum of $a_i(K)$. This completes the proof of the lemma.

We note that for the case of a diagonal matrix C the maximum of $|a_i(K)|$ was determined in /7/ in connection with a different physical problem.

If $\mathbf{p}_* = 0$, then $a_i^* = p_i(r_{ii} - \mathbf{r}_i^T R^{-1}\mathbf{r}_i)^{-1}$, which is identical with the corresponding value of a_i when $K=0$. In physical terms, this means that if the applied force contains one component, then application of non-conservative forces diminishes (does not increase) the corresponding oscillation amplitude.

Remark. These results enable us to solve the following problem, which is of independent interest in the theory of linear equations $A\mathbf{x} = \mathbf{p}$. Suppose that in the representation $A = R + K$, where R is symmetric and K skew-symmetric, the matrix R is of fixed sign. As is clear from the proof of the lemma, the extremal values of $a_i(K)$ are the roots of Eq.(3.12) Therefore, the solution of the system satisfies a two-sided estimate that is independent of the matrix K:

$$a_i^- \leqslant a_i \leqslant a_i^+, \quad a_i^- = c_i - \sqrt{c_i^2 + d_i}, \quad a_i^+ = c_i + \sqrt{c_i^2 + d_i}$$

If $p_* \neq 0$ (p_* is formed from p by deleting the i-th component), these become greatest lower and least upper bounds; they are attained when the elements of K satisfy (3.11), where $a_i = a_i^-$ and $a_i = a_i^+$, respectively. If $p_* = 0$, one of the quantities a_i^-, a_i^+, equal to zero, is not attained at a finite K ($a_i \to 0$ as $\|K\| \to \infty$).

We note, moreover, that if $p_* = 0$ our result implies that the absolute values of the diagonal elements of R^{-1} cannot be less than the corresponding values for the matrix $(R+K)^{-1}$.

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STUDY OF THE QUASILINEAR OSCILLATIONS OF MECHANICAL SYSTEMS WITH DISTRIBUTED AND LUMPED PARAMETERS*

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The averaging method is used to study a class of complex oscillatory systems which are described by vector integrodifferential equations with oscillating kernels. These equations arise when analysing mechanical objects which contain elements with distributed and lumped inertial and elastic parameters. Two physically distinct cases of the oscillation of rigid bodies are considered: "resonant" and "non-resonant", as determined by the properties of the mean values of the kernels of the integral terms. In the first case, it is shown that the equations of the first approximation are equivalent to a system of ordinary second-order differential equations, i.e., the order of the system of equations of the motion of a rigid body is doubled. In the second case, sufficient conditions are found for the oscillating initial variables to be slow in the usual sense of the averaging method; the order of the system is then preserved. The conditions are stated, under which the averaging method can be shown to be strictly applicable in asymptotically long time intervals and constructive error estimates are obtained. On the basis of this approach the perturbed horizontal oscillations of a rigid body containing a rectangular cavity with a two-layer heavy fluid which is elastically connected with a fixed base are investigated and qualitative effects are discovered and examined.

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